逻辑

1. 基本符号

$\neg, \wedge, \vee, \rightarrow, \leftrightarrow$

2. 基本等价命题

Some important equivalences:

☐ Identity laws	$p \vee F \Leftrightarrow p$	$p \wedge T \Leftrightarrow p$
☐ Domination laws	$p \vee T \Leftrightarrow T$	$p \wedge F \Leftrightarrow F$
☐ Idempotent laws	$p \vee p \Leftrightarrow p$	$p \wedge p \Leftrightarrow p$
☐ Complementation law	$\neg(\neg p) \Leftrightarrow p$	
☐ Commutative laws	$p \vee q \Leftrightarrow q \vee p$	$p \wedge q \Leftrightarrow q \wedge p$
☐ Associative laws	$p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$	
	$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$	
☐ Distributive laws	$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$	
	$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$	
☐ Laws of excluded middle	$p \wedge \neg p \Leftrightarrow F$	$p \vee \neg p \Leftrightarrow T$
☐ Absorption laws	$p \wedge (p \vee q) \Leftrightarrow p$	
	$p \vee (p \wedge q) \Leftrightarrow p$	
☐ De Morgan's laws	$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$	
	$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$	

3. 正则表达式（析取范式）DNF

$(A_{11} \wedge A_{12} \wedge \cdots A_{1m_1}) \vee \cdots \vee (A_{k1} \wedge A_{k2} \cdots \wedge A_{km_k})$

化DNF方法：① 直接化（$\neg / \leftrightarrow / \rightarrow$ 均可化为DNF）

② 根据真值表

※ 主析取范式（极小项之并）

怎么化DNF至FDF（$A \Leftrightarrow (A \wedge p) \vee (A \wedge \neg p)$）
（缺p）

4. 推理规则

• Rules of Inference

1. Law of detachment or modus ponens
   ☐ $p \rightarrow q$,
   ☐ p
   Therefore, q

2. Modus tollens
   ☐ $p \rightarrow q$,
   ☐ $\neg q$
   Therefore, $\neg p$

3. Rule of Addition
   ☐ p
   Therefore, $p \vee q$

4. Rule of simplification
   ☐ $p \wedge q$
   Therefore, p

5. Rule of conjunction
   ☐ p
   ☐ q
   Therefore, $p \wedge q$

6. Rule of hypothetical syllogism
   ☐ $p \rightarrow q$
   ☐ $q \rightarrow r$
   Therefore, $p \rightarrow r$

7. Rule of disjunctive syllogism
   ☐ $p \vee q$
   ☐ $\neg p$
   Therefore, q

反证法：

2) Another important proof technique is contradiction. The proof of $p \rightarrow q$ by contradiction consists of the following steps:

– Assume p is true and q is false
– Show that $\neg p$ is also true.
– Since the statement $p \wedge (\neg p)$ is always false. (Contradiction!)

$$((p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow q)^{(*)} \Leftrightarrow \neg(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \vee q$$
$$\Leftrightarrow \neg(p_1 \wedge p_2 \wedge \cdots \wedge p_n \wedge \neg q)^{(**)}$$

矛盾律

(*) is a tautology if and only if (**) is a contradiction.

5. 谓语（predicates） 量词（quantifiers）

↑

函数化的命题

全称量词 $\forall$

存在量词 $\exists$

1) When all the elements in the universe of discourse can be listed — say, $x_1, x_2, \cdots, x_n$, then

$$\forall x P(x) \Leftrightarrow P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n)$$
$$\exists x P(x) \Leftrightarrow P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n).$$

2) $\forall x \forall y p(x,y) \Leftrightarrow \forall y \forall x p(x,y)$ 同一量词可交换
$\exists x \exists y p(x,y) \Leftrightarrow \exists y \exists x p(x,y)$

But

$\forall x \exists y p(x,y) \not\Leftrightarrow \exists y \forall x p(x,y)$

Example: $\forall x \exists y (x+y=0)$ and $\exists y \forall x (x+y=0)$
– have different truth values.

一些等价关系

Some important Equivalent Predicates Formula

• De Morgan's laws for predicates:
$\neg \forall x p(x) \Leftrightarrow \exists x \neg p(x)$
$\neg \exists x p(x) \Leftrightarrow \forall x \neg p(x)$

• Quantifiers — handle with care!
$\forall x (p(x) \wedge q(x)) \Leftrightarrow (\forall x p(x)) \wedge (\forall x (q(x))$
$\exists x (p(x) \vee q(x)) \Leftrightarrow (\exists x p(x)) \vee (\exists x (q(x))$

But

量词多不是高

※ $\forall x (p(x) \vee q(x)) \not\Leftrightarrow \forall x p(x) \vee \forall x q(x)$ ("$\Leftarrow$" still holds)
$\exists x (p(x) \wedge q(x)) \not\Leftrightarrow \exists x p(x) \wedge \exists x (q(x)$ ("$\Rightarrow$" still holds)

两个x不一定相同

• More logical equivalence:
1) $A \wedge \forall x q(x) \Leftrightarrow \forall x (A \wedge q(x))$, x is not occurring in A
$A \vee \forall x q(x) \Leftrightarrow \forall x (A \vee q(x))$, x is not occurring in A
$\forall x q(x) \rightarrow A \Leftrightarrow \exists x (q(x) \rightarrow A)$, x is not occurring in A
$A \rightarrow \forall x q(x) \Leftrightarrow \forall x (A \rightarrow q(x))$, x is not occurring in A

2) $A \wedge \exists x q(x) \Leftrightarrow \exists x (A \wedge q(x))$, x is not occurring in A
$A \vee \exists x q(x) \Leftrightarrow \exists x (A \vee q(x))$, x is not occurring in A
$\exists x q(x) \rightarrow A \Leftrightarrow \forall x (q(x) \rightarrow A)$, x is not occurring in A
$A \rightarrow \exists x q(x) \Leftrightarrow \exists x (A \rightarrow q(x))$, x is not occurring in A

量词作条件要些的改变

前束范式

☐ PRENEX NORMAL FORMS

• Motivation:
– It simplifies the surface structure of the sentence.
– It is useful to automated theorem proving.

Definition: A formula is in prenex normal form if it is of the form

$$Q_1 x_1 Q_2 x_2 \cdots Q_n x_n B$$

量词

where $Q_i (i = 1, 2, \cdots, n)$ is $\forall$ or $\exists$ and B is quantifier free.

• Algorithm of prenex normal form
Any expression can be converted into prenex normal form by the following steps:

1. Rename bounded variables if necessary.
2. Eliminate all connectives $\rightarrow$ and $\leftrightarrow$.
3. Move all negations inward such that, in the end, negation only appear as part of literals.
4. Standardize the variables a part(when necessary).
5. move all quantifiers to the front of the formula.

一些推理方法：

☐ METHODS OF PROOF (Continued)
— for predicates

• Rules of Inference for quantifiers

1. Universal instantiation(UI)
   ☐ $\forall x \in D, P(x)$
   ☐ $d \in D$
   Therefore $P(d)$

2. Universal generalization(UG)
   ☐ $P(d)$ for any $d \in D$
   Therefore $\forall x P(x)$

3. Existential instantiation(EI)
   ☐ $\exists x \in D$
   ☐ $P(x)$
   Therefore $P(d)$ for some $d \in D$

4. Existential generalization(EG)
   ☐ $P(d)$ for some $d \in D$
   Therefore $\exists x P(x)$

集合

1. 集合的运算

$A \cup B \quad A \cap B \quad A - B = \{x \mid x \in A \wedge x \notin B\}$

$\overline{A} = U - A \quad A \oplus B = (A-B) \cup (B-A)$

☐ SET IDENTITIES

Identity laws	$A \cup \emptyset = A$	$A \cap U = A$
Domination laws	$A \cup U = U$	$A \cap \emptyset = \emptyset$
Idempotent laws	$A \cup A = A$	$A \cap A = A$
Complementation law	$\overline{\overline{A}} = A$	
Commutative laws	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Associative laws	$A \cup (B \cup C) = (A \cup B) \cup C$	
	$A \cap (B \cap C) = (A \cap B) \cap C$	
Distributive laws	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	
	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	
Laws of excluded middle	$A \cap \overline{A} = \emptyset$	$A \cup \overline{A} = U$
Absorption laws	$A \cap (A \cup B) = A$	
De Morgan's laws	$\overline{A \cup B} = \overline{A} \cap \overline{B}$	$\overline{A \cap B} = \overline{A} \cup \overline{B}$

2. 幂集（$P(S)$ 或 2^S）

3. 有限集与无限集的势

(1) 有限集 $|A \cup B| = |A| + |B| - |A \cap B|$

(2) 无限集 若A与B存在一个一一对应关系，则A与B等势

可数无限集（与自然数集一一对应）：整数、有理数、$\mathbb{N} \times \mathbb{N}$

不可数无限集

① 可数个可数无限集之并 可数

② $|2^A| > |A|$

③ 实数集不可数

0 $\quad 0.a_{00}a_{01} \cdots$

1 $\quad 0.a_{10}a_{11} \cdots$

2 $\quad 0.a_{20}a_{21}a_{22} \cdots$

构造 $b = 0.b_0 b_1 \cdots b_n \cdots$

$$b_i = \begin{cases} 1, & \text{if } a_{ii} \neq 1 \\ 2, & \text{if } a_{ii} = 1 \end{cases}$$

函数

1. 单射，满射，双射

2. O 符号 $\exists c, k$，对 $\forall x > k$ 有 $|f(x)| \leq c|g(x)|$

$\Rightarrow f(x) = O(g(x))$

Ω、Θ 类似

计数方法

1. n个相同球入m个不同盒子 $\rightarrow C_{m+n-1}^{n}$

2. 递推关系

Theorem Let $c_1, \cdots, c_k$ be real numbers. Suppose that the characteristic equation $r^k - c_1 r^{k-1} - \cdots - c_k = 0$ has t distinct roots $r_1, \cdots, r_t$ with multiplicities $m_1, \cdots, m_t$, respectively, so that $m_i \geq 1$ for $i = 1, 2, \cdots, t$ and $m_1 + m_2 + \cdots + m_t = k$. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ iff

$$a_n = (\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1} n^{m_1 - 1}) r_1^n$$
$$+ (\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2} n^{m_2 - 1}) r_2^n$$
$$+ \cdots + (\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t} n^{m_t - 1}) r_t^n$$

for $n = 0, 1, 2, \cdots$, where $\alpha_{i,j}$ are constants.

非线性

线性 { 齐解

Theorem: Suppose that $\{a_n\}$ satisfies the linear nonhomogeneous recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)$ where $c_1, \cdots, c_k$ are real numbers, and $F(n) = (b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0) s^n$, where $b_1, \cdots, b_t$ and s are real numbers.
1) When s is not a characteristic root of the associated linear homogeneous recurrence relation, there is a particular solution of the form
$$(p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$
2) When s is a characteristic root with multiplicity m, there is a particular solution of the form
$$n^m (p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$

3. 生成函数

Table 1 Useful Generating Functions.

$G(x)$	a_k
$(1+x)^n = \sum_{k=0}^{n} C(n,k) x^k$	$C(n,k)$
$(1+\alpha x)^n = \sum_{k=0}^{n} C(n,k) \alpha^k x^k$	$C(n,k)\alpha^k$
$(1+x^r)^n$	$C(n,k/r)$ if $r \mid k$; 0 otherwise
$\frac{1-x^{n+1}}{1-x} = \sum_{k=0}^{n} x^k$	1 if $k \leq n$; 0 otherwise
$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$	1
$\frac{1}{1-\alpha x} = \sum_{k=0}^{\infty} \alpha^k x^k$	α^k
$\frac{1}{1-x^r} = \sum_{k=0}^{\infty} x^{rk}$	1 if $r \mid k$; 0 otherwise
$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1) x^k$	$k+1$
※ $\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k) x^k$	$C(n+k+1,k)$
$\frac{1}{(1+x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)(-x)^k$	$(-1)^k C(n+k+1,k)$
$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$	$1/k!$

常用

指数型生成函数：$\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$

$$\binom{u}{k} = \binom{k+u-1}{k} \cdot (-1)^k$$

当 $u < 0$ 时

关系

1. { reflexive / irreflexive

{ symmetric / antisymmetric

transitive

2. 求传递关系的闭包

① $R^* = \bigcup_{i=1}^{n} R^i$

（R^n 理解为矩阵乘法）

② WARSHALL 算法，中间点一个个加

$$W_0 = \begin{bmatrix} 0&0&0&1 \\ 0&0&1&0 \\ 1&0&0&1 \\ 0&0&1&0 \end{bmatrix} \quad W_1 = W_2 = \begin{bmatrix} 0&0&0&1 \\ 1&0&1&1 \\ 0&0&0&0 \\ 0&0&1&0 \end{bmatrix}$$

$$W_3 = \begin{bmatrix} 0&0&0&1 \\ 1&0&1&1 \\ 1&0&0&1 \\ 1&0&1&1 \end{bmatrix} \quad W_4 = \begin{bmatrix} 1&0&1&1 \\ 1&0&1&1 \\ 1&0&1&1 \\ 1&0&1&1 \end{bmatrix}$$

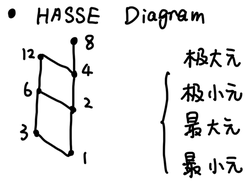
3. 等价关系 reflexive / symmetric / transitive

4. 分划 partition

5. 偏序 reflexive antisymmetric transitive

全部都可比较的偏序集为全序集

• HASSE Diagram



极大元
极小元
最大元
最小元

• 良序集（全序+最小元）（≤）

• ☐ LATTICES

Definition: A partially ordered set in which pair of elements has both a least upper bound and a greatest lower bound is called **lattice**.

Example: 1) Is the poset $(\mathbb{Z}, |)$ a lattice?
2) Is the poset $(\{1,2,3,4,5\}, |)$ a lattice?
3) Determine whether $(2^S, \subseteq)$ is a lattice whether $S = \{a,b,c\}$.

• ☐ MAXIMAL AND MINIMAL ELEMRNTS

Definition: Let $(A, \preceq)$ be a partial ordered set, $B \subseteq A$.
1) a is a **maximal element**（极大元）of B: $a \in B \wedge$ There is no $x \in B$ such that $a \prec x$.
b is a **minimal element**（极小元）of B: $b \in B \wedge$ There is no $x \in B$ such that $x \prec b$.
2) a is the **greatest element**（最大元）of B: $a \in B \wedge \forall x \in B (x \preceq a)$.
b is the **least element**（最小元）of B: $b \in B \wedge \forall x \in B (b \preceq x)$.

02:24:55

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Definition: (continued)
3) c is an **upper bound** of B: $c \in A \wedge \forall x \in B : x \preceq c$.
$d \in A$ is an **lower bound** of B : $d \in A \wedge \forall x \in B : d \preceq x$.
4) c is an **least upper bound** of B: c is a upper bound of B and $\forall x$ is a upper bound of $B \Rightarrow c \preceq x$.
$d \in A$ is an **greatest lower bound** of B: d is a lower bound of B and $\forall x$is a lower bound of $B \Rightarrow x \preceq d$.

图

1.

Table 1 GRAPH TERMINOLOGY.

Type	Edges	Multiple Edges Allowed?	Loop Allowed?
Simple Graph	Undirected	No	No
Multigraph	Undirected	Yes	No
Pseudograph	Undirected	Yes	Yes
Directed graph	Directed	No	Yes
Directed multigraph	Directed	Yes	Yes

2. 一些特殊的图

(1) K_n 完全图

(2) C_n 圈

(3) W_n 轮

(4) n-cubes

(5) 二部图（Bipartite graph）

3. 相关矩阵

(1) 邻接矩阵（adjacent）

(2) 关联矩阵（incidence）

• Incidence Matrix(关联矩阵)

Definition: Let $G = (V, E)$ be an undirected graph. Suppose that $v_1, v_2, \cdots, v_n$ are the vertices and $e_1, e_2, \cdots, e_m$ are the edges. Then the **incidence matrix** with respect to this ordering of V and E is matrix $M = [m_{ij}]_{n \times m}$, where

$$m_{ij} = \begin{cases} 1 & \text{when edge } e_j \text{ is incidence with } v_i, \\ 0 & \text{otherwise.} \end{cases}$$

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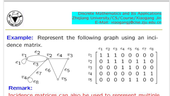
Example: Represent the following graph using an incidence matrix.



$$\begin{array}{c|ccccccccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} & e_{11} \\ \hline v_1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ v_2 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ v_3 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ v_4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ v_5 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ v_6 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{array}$$

Remark:
Incidence matrices can also be used to represent multiple edges and loops.
1) Multiple edges ⇔ columns with identical entries.
2) Loops ⇔ columns with exactly one entry equal to 1.

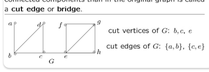
4. 图的同构

5. 图的连通性

• 连通图中任意两点 均存在 简单通路（不含重复边）

• （无向）

Definition: The removal of a vertex and all edges incident with it produces a subgraph with more connected components than in the original graph. Such vertices are called **cut vertices**.
An edge whose removal produces a subgraph with more connected components than in the original graph is called a **cut edge** or **bridge**.



cut vertices of G: b, c, e

cut edges of G: $\{a,b\}, \{c,e\}$

• 有向图的强连接与弱连接

• 计算两个点之间的路径数

长度为r ⇒ A^r （$A^{r+1} = A^r \cdot A$）

6. 欧拉路与哈密顿路

(1) 欧拉回路（连通，每个点均偶数度） 过每条边

欧拉路径（可以有两个奇数点），

欧拉图（包含欧拉回路）

(2) 哈密顿路 过每个点，

哈密顿回路

哈密顿图

充分条件：任意两点度数之和 ≥ n，则该图为哈密顿图

7. 最短路径问题

• Dijkstra 算法

8. 平面图

• 欧拉公式 $V - e + r = 1 + k$（递推公式）

推论：G为平面图（$V \geq 3$）⇒ $e \leq 3V - 6$

• **Theorem:** (Kuratowski's theorem (1930))
A graph G is nonplanar if and only if G contains a subgraph homeomorphic to K_5 or $K_{3,3}$.